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ERROR OF APPROXIMATION OF FUNCTION IN LIPSCHITZ CLASS VIA ALMOST (N, p_k, q_k)-MEANS*

Kusum Sharma[1], Diksha Dumka[1], Laxmi Rathore[2], M. K. Sharma[3], Lakshmi Narayan Mishra[4†]

[1]Department of Mathematics, National Institute of Technology Uttarakhand, Srinagar Garhwal- 246174, Uttarakhand, India
[2]Ward Number 16, Bhagatbandh, Anuppur 484 224, Madhya Pradesh, India
[3]Department of Mathematics, C.C.S. University, Meerut-250004, India
[4]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India

Abstract. This paper is aimed to establish a new result on error of approximation of function in Lipschitz $Lip\,(\rho\,(t)\,,r)(r \geq 1)$- class via almost (N,$p_k$,$q_k$)-means using the concept of head bounded variation sequences (HBVS) and rest bounded variation sequences (RBVS) of Fourier series.

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[†]**Corresponding author:** Lakshmi Narayan Mishra, Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India, e-mail: *lakshminarayanmishra04@gmail.com*
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1 Introduction

The summability of divergent series was studied by various researchers like Cesáro, Nörlund, Hardy, etc. But in 1948, it was Lorentz (1948) who first gave the idea of weak limit which further defined the concept of Almost Convergence and these summability methods are of three types - ordinary, absolute and strong. Following the similar lines, almost convergence of sequences are also of three types:

1. Strongly almost convergence
2. Absolutely almost convergence
3. Almost convergence

These concepts of strong and absolutely almost convergence were discussed in Das et al. (1984); King (1966); Maddox (1978). The error of approximation of functions belonging to the Lipschitz class of the form $Lip\alpha$, $Lip(\rho(t), r)$ and $W(L_r, \rho(t))$ using single and double summability was studied by various researchers Khan (1973); Nigam & Sharma (2011a,b); Qureshi (1982). Later, literature flooded on the study of these Lipschitz classes using almost convergence specifically

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by almost Riesz means of its Fourier series by several mathematicians like Mishra et al. (2014), Deepmala & Piscoran (2016), Qureshi (1982), Sharma & Qureshi (2020) and Sharma (2020). This motivated us to study the error of approximation of function in $Lip(\rho(t), r)$ class via almost (N, p_k, q_k) means which in turn generalizes the study of Krasniqi (2015) for more general class.

Let h be a periodic function with period 2π and integrable in the sense of Lebesgue in the range $0 \leq x \leq 2\pi$. The trigonometric Fourier series of the function h is given by

$$h \sim \frac{a_0}{2} + \sum_{i=1}^{\infty} (a_i \cos ix + b_i \sin ix) \tag{1}$$

with $t_k(h; x)$ which is the $(k+1)^{th}$ partial sum also known as trigonometric polynomial of Fourier series.

Let $\{t_k\}$ be the sequence of partial sum corresponding to the infinite series $\sum_{k=0}^{\infty} x_k$. Let $p = \{p_k\}$ and $q = \{q_k\}$ be two given sequences defined as

$$P_k = p_0 + p_1 + ... + p_k; P_{-1} = p_{-1} = 0,$$

$$Q_k = q_0 + q_1 + ... + q_k; Q_{-1} = q_{-1} = 0,$$

and their convolution R_k is defined as

$$R_k = (p * q)_k = \sum_{\nu=0}^{k} p_{k-\nu} q_\nu \text{ so that } R_k \neq 0, \forall k. \tag{2}$$

Firstly, we try to establish some of the basics and important definitions which will be used in this paper.

1.1 Definition

A sequence $\{x_k\}$ is said to be almost convergent to a finite number s if Lorentz (1948)

$$\lim_{k \to \infty} s_{k,j} = \lim_{k \to \infty} \frac{1}{n+1} \sum_{\mu=j}^{k+j} x_\mu = s \text{ uniformly with respect to } j.$$

1.2 Definition

The Fourier series (1) of the function h is said to be almost Nörlund summable to a finite limit s Borwein (1958) if

$$N_{k,j}^{p,q} = \frac{1}{R_k} \sum_{\nu=0}^{k} p_{k-\nu} q_\nu s_{\nu,j} \to s, \text{as}\, k \to \infty, \tag{3}$$

uniformly with respect to j, where

$$s_{\nu,j} = \frac{1}{\nu+1} \sum_{\mu=j}^{\nu+j} s_\mu. \tag{4}$$

Almost generalized Nörlund method is said to be regular if $\sum_{\nu=0}^{k} |p_{k-\nu} q_\nu| = O(|R_k|)$ and $p_{k-\nu} = o(R_k)$ as $k \to \infty$, for every fixed $\nu \geq 0$, (Hardy, 1949).

Remark: A few particular cases of generalized Nörlund method are given below:

1. (N, p_k, q_k) method is converted in to (N, p_k) if $q_j = 1$; $\forall\, k$.

2. (N, p_k, q_k) method is converted to Riesz method i.e. $\left(\bar{N}, q_k\right)$ if $p_k = 1; \forall\, k$.

3. Taking $p_k = \begin{pmatrix} k+\beta-1 \\ \beta-1 \end{pmatrix}$, $\beta > 0$, then this method is reduced into Cesàro method i.e. (C, β).

4. Again taking $p_k = \frac{1}{k+1}$ in Nörlund method,it will be reduced to $\left(N, \frac{1}{k+1}\right)$ i.e. Harmonic mean.

Now we define the RBVS and HBVS (Leindler, 2001) as follows:

1.3 Definition

Let consider a sequence of positive numbers having only finite non-zero terms i.e. $\mathbf{b} = \{b_k\}$, then it is known as head bounded variation sequence if

$$\sum_{r=0}^{k-1} |b_r - b_{r+1}| \leq \lambda\left(\mathbf{b}\right) b_k, \text{for all}\, k \in \mathbb{N}\, \text{or}\, k \leq \mathbb{N} \tag{5}$$

and b_N is the last non-zero term of the sequence $\{b_k\}$.

1.4 Definition

A sequence $\mathbf{b} = \{b_k\}$ is known as rest bounded variation sequence if

$$\sum_{r=k}^{\infty} |b_r - b_{r+1}| \leq \lambda\left(\mathbf{b}\right) b_k, \text{for all}\, k \in \mathbb{N}, \tag{6}$$

where $\lambda\left(\mathbf{b}\right)$ is a constant which depends only on $\mathbf{b}$. RBVS is more general than that of monotone sequences (Zhou et al., 2010).

Now we state the Hölder's inequality for integrals which is given by

$$\|gh\|_1 = \|g\|_p \|h\|_q \ \text{ for } \frac{1}{p} + \frac{1}{q} = 1\ , p \geq 1,$$

where g and h are continuous functions on $[a, b]$. Here $\|g\|_p$ is defined as

$$\|g\|_p = \left(\int_a^b |g\left(x\right)|^p\, dx\right)^{1/p}\ ;\ 1 \leq p < \infty.$$

We say a function h belongs to the class $Lip\,\alpha$ if $|h\left(x+t\right) - h\left(x\right)| = \mathcal{O}\left(|t|^{\alpha}\right)$, for $0 < \alpha \leq 1$ and $h \in Lip\left(\alpha, r\right)\left(r \geq 1\ \right)$ -class, if
$\left\{\int_a^b |h\left(x+t\right) - h\left(x\right)|^r\, dx\right\}^{\frac{1}{r}} = \mathcal{O}\left(|t|^{\alpha}\right), a \leq x \leq b,\, 0 < \alpha \leq 1$ (McFadden, 1978).
For $a \leq x \leq b$, $0 < \alpha \leq 1$, we say a function h belongs to $Lip\left(\rho\left(t\right), r\right)(r \geq 1)$-class if

$$\left\{\int_a^b |h\left(x+t\right) - h\left(x\right)|^r\, dx\right\}^{\frac{1}{r}} = \mathcal{O}\left(\rho(t)\right)).$$

It is easy to see that $Lip\left(\rho\left(t\right), r\right) \equiv\ Lip\left(\alpha, r\right)$ if we take $\rho\left(t\right) = |t|^{\alpha}$ and also $Lip\left(\alpha, r\right) \equiv Lip\alpha$ class if $r \to \infty$. Hence $Lip\left(\rho\left(t\right), r\right)$ class is more general class.

For a function $h : \mathbb{R} \to \mathbb{R}$, the L_r-norm of h is $\|h\|_r = \left(\int_0^{2\pi} |h\left(x\right)|^r\, dx\right)^{\frac{1}{r}}$, $r \geq 1$, and the L_{∞}-norm is $\|h\|_{\infty} = \sup_{x\in[0,2\pi]} \left\{|h\left(x\right)| : x \in \mathbb{R}\right\}$.

Let $\phi(x)$ and $\xi(x)$ both be integrable in $[a,b]$ and $\xi(x)$ is monotonic in the given range. Then, $\exists\ \tau \in (a,b)$ such that

$$\int_a^b \phi(x)\,\xi(x)\,dx = \xi(a)\int_a^\tau \phi(x)\,dx + \xi(b)\int_\tau^b \phi(x)\,dx.$$

The above theorem is known as second mean value theorem for integrals.

The error of approximation $E_k(h)$ of a function $h:\mathbb{R}\to\mathbb{R}$ by trigonometric polynomial $t_k(x)$ is defined by

$$E_k(h) = \|t_k - h\|_\infty, \tag{7}$$

where $E_n(h)$ is defined under sup norm $\|.\|_\infty$ (Zugmund, 1968) and $E_n(h) = \min_x \|t_n - h\|_r$ if $h \in L^r$. This method of approximation is Trigonometric Fourier Approximation (TFA).

2 Known Theorem

Krasniqi (2015) proved the following theorem:

Theorem 1. *Let $h:\mathbb{R}\to\mathbb{R}$ be a 2π periodic function, Lebesgue integrable and belonging to the $Lip(\alpha,r),(r\ge 1)$ class and let $(p_k)\in HBVS$ and $(q_k)\in RBVS$, then the degree of approximation of the function hby almost generalized Nörlund means of its Fourier series $t_{k,j}^{p,q}(h(t);x)$ is given*

$$\left\|h(t) - t_{k,j}^{p,q}(h(t);x)\right\|_r = \mathcal{O}\left(R_k^{1/r-\alpha}\right), \forall k,$$

provided $\psi(t)$ satisfies the following conditions

$$\left[\int_0^{\pi/R_k}\left(\frac{t\,|\psi(t)|}{t^\alpha}\right)^r dt\right]^{1/r} = \mathcal{O}\left(R_k^{-1}\right), \tag{8}$$

$$\left[\int_{\pi/R_k}^{\pi}\left(\frac{t^{-\delta}\,|\psi(t)|}{t^\alpha}\right)^r dt\right]^{1/r} = \mathcal{O}\left(R_k^{\delta}\right), \tag{9}$$

where δ is a finite quantity, and $r+s=rs$ for $1\le r\le\infty$.

3 Main Theorem

In this present work, we prove the following theorem:

Theorem 2. *Let $h:\mathbb{R}\to\mathbb{R}$ be a 2π-periodic function (Lebesgue integrable) belonging to $Lip(\rho(t),r)\,(r\ge 1)$-class and let $p=\{p_n\}$ and $q=\{q_n\}$ are HBVS and RBVS respectively. Then the error of approximation of h via almost (N,p_k,q_k) means of its Fourier series (1) is given*

$$\left\|N_{k,j}^{p,q} - h\right\|_r = \mathcal{O}\left(\rho\left(\frac{1}{R_k}\right)R_k^{1/r}\right), \forall k, \tag{10}$$

where $\rho(t)$ is increasing function and satisfies the following conditions:

$$\frac{\rho(t)}{t} \textit{ is decreasing function in } t; \tag{11}$$

$$\left[\int_0^{\pi/R_k}\left(\frac{t\,|\psi(t)|}{\rho(t)}\right)^r dt\right]^{1/r} = \mathcal{O}\left(R_k^{-1}\right); \tag{12}$$

$$\left[\int_{\pi/R_k}^{\pi}\left(\frac{t^{-\delta}\left|\psi\left(t\right)\right|}{\rho(t)}\right)^r dt\right]^{1/r}=\mathcal{O}\left(R_k^{\delta}\right), \tag{13}$$

where $N_{k,j}^{p,q}$ is almost Nörlund means defined in equation (3), δ is an arbitrary number, and $r+s=rs$, $1\leq r\leq\infty$ and the equations (12) and (13) hold uniformly in x.

Proof. Following Titchmarsh (1939) and using Riemann-Lebesgue theorem, the partial sums $s_{k,j}\left(h\left(t\right);x\right)$ is given by

$$s_{k,j}\left(h\left(t\right);x\right)=\frac{1}{2\pi\left(i+1\right)}\int_0^{\pi}\psi\left(t\right)\frac{\cos pt-\cos\left(i+p+1\right)}{2\sin^2\frac{t}{2}}dt.$$

Therefore using $N_{k,j}^{p,q}\left(h\left(t\right);x\right)$ for almost generalized Nörlund means of $s_{k,j}\left(h\left(t\right);x\right)$; we have

$$\begin{aligned}
&\left|N_{k,j}^{p,q}-h\right|\\
&=\left|\frac{1}{R_k}\sum_{i=0}^{k}p_k q_{k-i}\left\{s_{k,j}\left(h\left(t\right);x\right)-h\left(t\right)\right\}\right|\\
&\leq\frac{1}{2\pi R_k}\int_0^{\pi}\left|\psi\left(t\right)\right|\left|\sum_{i=0}^{k}\frac{p_i q_{k-i}}{i+1}.\frac{\cos pt-\cos\left(i+p+1\right)}{2\sin^2\frac{t}{2}}\right|dt\\
&\leq\frac{1}{2\pi R_k}\left(\int_0^{\frac{\pi}{R_k}}+\int_{\frac{\pi}{R_k}}^{\pi}\right)\left|\psi\left(t\right)\right|\left|\sum_{i=0}^{k}\frac{p_i q_{k-i}}{i+1}\frac{\cos pt-\cos\left(i+p+1\right)}{2\sin^2\frac{t}{2}}\right|dt\\
&=\frac{1}{2\pi R_k}\left(\int_0^{\frac{\pi}{R_k}}+\int_{\frac{\pi}{R_k}}^{\pi}\right)\left|\psi\left(t\right)\right|.\\
&\quad\left|\sum_{i=0}^{k}\frac{p_i q_{k-i}}{i+1}\frac{\sin\left(i+2p+1\right)\frac{t}{2}.\sin\left(i+1\right)\frac{t}{2}}{\sin^2\frac{t}{2}}\right|dt\\
&=I_1+I_2\quad\text{(suppose)}
\end{aligned}\tag{14}$$

In order to evaluate I_1 consider

$$\left|I_1\right|\leq\frac{1}{2\pi R_k}\int_0^{\frac{\pi}{R_k}}\left|\psi\left(t\right)\right|\left|\sum_{i=0}^{k}\frac{p_i q_{k-i}}{i+1}\frac{\sin\left(i+2p+1\right)\frac{t}{2}.\sin\left(i+1\right)\frac{t}{2}}{\sin^2\frac{t}{2}}\right|dt.$$

Using Hölder's inequality, $\psi\left(t\right)\in\ Lip\left(\rho(t),r\right)$ - class and the inequality

$$\left|\sin kt\right|\leq k\left|\sin t\right|;\ \forall\, t\in\mathbb{R},\ k\in\mathbb{N} \tag{15}$$

we have

$$\begin{aligned}
\left|I_1\right|&\leq\frac{1}{2\pi R_k}\left(\int_0^{\frac{\pi}{R_k}}\left(\frac{t\left|\psi\left(t\right)\right|}{\rho\left(t\right)}\right)^r dt\right)^{1/r}\times\\
&\left(\int_0^{\frac{\pi}{R_k}}\left(\frac{\rho\left(t\right)}{t}\left|\sum_{i=0}^{k}\frac{p_i q_{k-i}}{i+1}\frac{\sin\left(i+2p+1\right)\frac{t}{2}.\sin\left(i+1\right)\frac{t}{2}}{\sin^2\frac{t}{2}}\right|\right)^s dt\right)^{1/s}\\
&=\mathcal{O}\left(R_k^{-2}\right)\left(\int_0^{\frac{\pi}{R_k}}\left(\frac{\rho(t)}{t}\left|\sum_{i=0}^{k}\frac{p_i q_{k-i}}{k+1}\frac{\left(i+1\right)\sin\left(i+2p+1\right)\frac{t}{2}}{\sin\frac{t}{2}}\right|\right)^s dt\right)^{1/s},\ \text{using equation (12)}\\
&=\mathcal{O}\left(R_k^{-2}\right)\left(\int_0^{\frac{\pi}{R_k}}\left(\frac{\rho(t)}{t^2}\left|\sum_{i=0}^{k}p_i q_{k-i}\right|\right)^s dt\right)^{1/s},\ \text{using inequality}\left(\sin\frac{t}{2}\right)\leq\frac{t}{\pi},\ \ 0<t\leq\pi\\
&=\mathcal{O}\left(R_k^{-2}\right)\left(\int_0^{\frac{\pi}{R_k}}R_k^s\left(\frac{\rho(t)}{t^2}\right)^s dt\right)^{1/s}.
\end{aligned}$$

In view of $\rho(t)$ is increasing function in t, applying second mean value theorem for integrals, one obtains

$$\begin{aligned}
&= \mathcal{O}\left(R_k^{-1}\rho\left(\frac{1}{R_k}\right)\right)\left(\lim_{\in\to 0}\int_{\in}^{\frac{\pi}{R_k}} t^{-2s}dt\right)^{1/s} \\
&= \mathcal{O}\left(R_k^{-1}\rho\left(\frac{1}{R_k}\right)\right)\left(\frac{t^{-2+\frac{1}{s}}}{-2+\frac{1}{s}}\right)_{\in}^{\frac{\pi}{R_k}} \\
&= \mathcal{O}\left(R_k^{-1}\rho\left(\frac{1}{R_k}\right)R_k^{2-\frac{1}{s}}\right) \\
&= \mathcal{O}\left(R_k^{\frac{1}{r}}\rho\left(\frac{1}{R_k}\right)\right) \qquad \text{since } r+s=rs;\ 1\le r<\infty.
\end{aligned} \tag{16}$$

Furthermore, considering I_2 and applying Hölder's inequality, one gets

$$|I_2| \le \frac{1}{2\pi R_k}\left(\int_{\frac{\pi}{R_k}}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\rho(t)}\right)^r dt\right)^{1/r}\times$$
$$\left(\int_{\frac{\pi}{R_k}}^{\pi}\left(\frac{\rho(t)}{t^{-\delta}}\left|\sum_{i=0}^{k}\frac{p_i q_{k-i}}{i+1}\frac{\sin(i+2p+1)\frac{t}{2}.\sin(i+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\right|\right)^s dt\right)^{1/s}.$$

Again using the fact that $\psi(t)\in Lip(\rho(t),r)$, conditions (13) and (15) and inequality $\left(\sin\frac{t}{2}\right)\le\frac{t}{\pi},\ \ 0<t\le\pi$, we get

$$|I_2| \le \mathcal{O}(R_k^{\delta-1})\left(\int_{\frac{\pi}{R_k}}^{\pi}\left(\frac{\rho(t)}{t^{-\delta+1}}\left|\frac{p_i q_{k-i}}{i+1}(i+1)\sin(i+2p+1)\frac{t}{2}\right|\right)^s dt\right)^{1/s}$$

$$= \mathcal{O}\left(R_k^{\delta-1}\right)\int_{\frac{\pi}{R_k}}^{\pi}\left(\frac{\rho(t)}{t^{-\delta+1}}\left|\sum_{i=0}^{k}p_i q_{k-i}\right|^s dt\right)^{1/s},\ \text{since } |\sin t|\le 1 \tag{17}$$

By the hypothesis, $\{p_k\}$ is a head bounded variation sequence, therefore by (5), we obtain

$$\begin{aligned}
p_m - p_k \le |p_m - p_k| &\le \sum_{i=m}^{k-1}|p_i - p_{i+1}| \\
&\le \lambda(p)\,p_k \\
p_m &\le \lambda(p)\,p_k + p_k = (\lambda(p)+1)\,p_k,\ \forall m\in[0,k].
\end{aligned}$$

Also, using equation (6) as $\{q_k\}$ is RBVS, we get

$$q_{k-m}\le\lambda(q)\,q_k,\ \forall m\in[0,k]$$

From the above equations, we obtain

$$\sum_{i=0}^{k}|p_i q_{k-i}| \le \sum_{i=0}^{k}(\lambda(p)+1)\,p_k\lambda(q)\,q_k \tag{18}$$

Since $R_k=\sum_{\nu=0}^{k}p_{k-\nu}q_\nu$, thus we obtain

$$(\lambda(p)+1)\,p_k\lambda(q)\,q_k = \mathcal{O}(R_k) \tag{19}$$

Using above equation in (17), we get

$$|I_2| = \mathcal{O}\left(R_k^{\delta}\right)\left(\int_{\frac{\pi}{R_k}}^{\pi}\left(\frac{\rho(t)}{t^{-\delta+1}}\right)^s dt\right)^{1/s}.$$

From this taking $t = \frac{1}{\mu} \Rightarrow dt = -\frac{1}{\mu^2}d\mu$,

$$|I_2| = \mathcal{O}\left(R_k^{\delta}\right)\left(\int_{\frac{1}{\pi}}^{\frac{R_k}{\pi}}\left(\frac{\rho\left(\frac{1}{\mu}\right)}{\frac{1}{\mu}.\mu^{\delta}}\right)^s \frac{d\mu}{\mu^2}\right)^{1/s}.$$

In view of $\frac{\rho(t)}{t}$ is a decreasing function, we have $\frac{\rho\left(\frac{1}{\mu}\right)}{\frac{1}{\mu}}$ as increasing. Therefore using the second mean value theorem for integrals, we get

$$\begin{aligned}
&= \mathcal{O}\left(R_k^{\delta}.\, R_k\, \rho\left(\frac{1}{R_k}\right)\right)\left(\int_{\frac{1}{\pi}}^{\frac{R_k}{\pi}} \mu^{-\delta s-2}d\mu\right)^{1/s}\\
&= \mathcal{O}\left(R_k^{\delta+1}.\, \rho\left(\frac{1}{R_k}\right)\right).\left(\frac{\mu^{-\delta-\frac{1}{s}}}{-\delta-\frac{1}{s}}\right)_{\frac{1}{\pi}}^{\frac{R_k}{\pi}}\\
&= \mathcal{O}\left(R_k^{\delta+1}.\rho\left(\frac{1}{R_k}\right).R_k^{-\delta-\frac{1}{s}}\right)\\
&= \mathcal{O}\left(\rho\left(\frac{1}{R_k}\right).R_k^{\frac{1}{r}}\right) \qquad (20)
\end{aligned}$$

Combining (14), (16) and (20), one can get the required result.

$$I = \left\|N_{k,j}^{p,q} - h(t)\right\| = \mathcal{O}\left(\rho\left(\frac{1}{R_k}\right).R_k^{\frac{1}{r}}\right)$$

□

4 Consequences

Error analysis of functions has become a varied field of research in recent years. The well known theorem of Weierstrass which is known as Weierstrass approximation theorem is the root of approximation theory. Approximation theory has been a major thrust of research with numerous applications in various fields via summability methods through trigonometric Fourier approximation. One of the important application of approximation theory via summability methods is improving the quality of digital filters and can be seen in the work of Psarakis & Moustakides (1997).
Two important corollaries can be deduced from our main result:

Corollary 1. *If take $\rho(t) = t^{\alpha}, 0 < \alpha < 1$, in main result, then $h \in Lip(\alpha, r)$ and*

$$\left|N_{k,j}^{p,q} - h\right| = \mathcal{O}\left\{R_k^{-\alpha+\frac{1}{r}}\right\}$$

which is the result of Krasniqi (2015).

Corollary 2. *If $\rho(t) = t^{\alpha}, 0 < \alpha < 1$, and $r \to \infty$, then $h \in Lip\alpha$ and*

$$\left|N_{k,j}^{p,q} - h\right| = \mathcal{O}\left\{R_k^{-\alpha}\right\}.$$

Remark 1: Both the above corollaries are true for almost (N, p_k, q_k) means.

Remark 2: Theorem 2 may be used in signal processing to analyze the behaviour of input data and changes in the process and through which the output can be made stable and bounded.

5 Conclusion

In this present work, the error of approximation of function in Lipschitz class by almost generalized Nörlund means using the method of HBVS and RBVS has been explained. This motivates the engineers, investigators working in the field of approximation theory and that are interested in the study of function through trigonometric polynomial. Validation of the Theorem 2 is done by corollary 1 which is the result established by Krasniqi (2015), hence a particular case of Theorem 2. The result obtained in this paper is more general than reported in the literature, thus enrich the literature of the field. Our result can be extended or generalized for weighted Lipschitz and zygmund classes in future.

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